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A CONJECTURE OF ERDÖS ON 3-POWERFUL NUMBERS

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ABSTRACT. Erdös conjectured that the Diophantine equation $x + y = z$ has infinitely many solutions in pairwise coprime 3-powerful integers, i.e., positive integers n for which $p \mid n$ implies $p^3 \mid n$. This was recently proved by Nitaj who, however, was unable to verify the further conjecture that this could be done infinitely often with integers x, y and z none of which is a perfect cube. This is now demonstrated.

Theorem. *The conjecture will follow if even one such solution can be found.*

Proof. Let $a + b = c$ be one solution. Then for these values of a, b and c the Diophantine equation $aX^3+bY^3 = cZ^3$ has the solution $[1, 1, 1]$ in integers $[X, Y, Z]$ with aX, bY and cZ pairwise coprime and hence has infinitely many. This is a special case of a well-known theorem [2], but is easily proved for our case. For starting from one such solution $[X, Y, Z]$, we find that another is given by $X' = X(bY^3 + cZ^3)$, $Y' = -Y(aX^3 + cZ^3)$, $Z' = Z(aX^3 - bY^3)$, as is easily verified. Here we find that $(a, Y') = (a, Y(aX^3 + cZ^3)) = (a, cYZ^3) = 1$ and so aX', bY' and cZ' are pairwise coprime provided X' and Y' are; this is not always true, for we find that $(X', Y') = (bY^3 + cZ^3, aX^3 + cZ^3) = (aX^3 + 2bY^3, 2aX^3 + bY^3) = 3$ or 1 according as 3 does or does not divide $aX^3 - bY^3$. However, dividing out by this common factor if it occurs, we obtain a new solution with $|X'Y'Z'| \geq 2|XYZ|$, for since the three nonzero integers X'/X, Y'/Y and Z'/Z have sum zero, their product must be at least 2 in absolute value.

For any such solution, $x = aX^3$, $y = bY^3$ and $z = cZ^3$ provides a solution of the original equation, and none of x, y and z will be a cube if none of a, b and c is.

The result therefore follows on observing that

$$\begin{aligned} X &= 97122476847715066049634904442 81, \\ Y &= 32295800804958334401937923416351, \\ Z &= 27474621855216870941749052236511, \end{aligned}$$

is a solution of the equation $32X^3 + 49Y^3 = 81Z^3$, for which $7 \mid Y$. ∎

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